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COUPLED CREEP-ELASTOPLASTIC-DAMAGE ANALYSIS FOR ISOTROPIC AND ANISOTROPIC NONLINEAR MATERIALS

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Abstract—This paper deals with the computational aspects of a coupled creep-elastoplastic-damage analysis for anisotropic, and as a special case isotropic nonlinear materials. A three phase backward Euler integration algorithm for stress update is proposed. For anisotropic nonlinear materials a general direct stress return mapping algorithm, utilising Newton–Raphson iteration, is derived. The stress vector and scalar variables quantifying the incremental creep, plasticity and damage are updated simultaneously. For isotropic materials the elasto-plastic stress update algorithm for plane stress by Jetteur (1986, *Engng Comp.* **3**, 251–253) is extended to include creep and damage. In addition, a simple stress algorithm for the general three-dimensional isotropic case is also presented. The resulting algorithms are suitable for inclusion in general structural analysis codes. The consistent tangent matrix is also formulated for use in a global Newton iterative procedure, in which structural displacements are sought as the problem unknowns. Examples are given using the general purpose code LUSAS in which the algorithms have been implemented.

1. INTRODUCTION

Although this paper covers the combined integration of creep, damage and plasticity attention is focused on the damage process, since its inclusion is an extension to earlier work covering combined creep and plastic behaviour presented by Lyons *et al.* (1992).

The phenomenon of initiation and growth of cavities and microcracks in a material subjected to external forces is called "damage". Since the pioneering works of Kachanov (1958) and Rabotnov (1963) a new concept, called "continuum damage mechanics", has been introduced and widely accepted to describe such progressive material degradation (material damage). It is assumed that a reduction in the net area due to growth of cavities is the principal driving mechanism.

Continuum damage mechanics is based on the thermodynamics of an irreversible process (Chaboche, 1988a,b; Lemaitre, 1985; Bazant, 1988) and the internal state variable theory. Simo and Ju (1987a,b) have shown that material behaves isotropically for the case of ductile damage. However, for anisotropic damage a damage tensor must be utilized. In the following, attention will be restricted to the isotropic damage case for which it suffices to consider an internal scalar damage variable d. The value $d = 0$ corresponds to the undamaged state, whereas a value $d(t) \in (0, d_c)$ corresponds to a damaged state, with the value $d(t) = d_c$ defining complete local rupture. d_c is a given constant and can be assumed to lie in the range $0.2 \leqslant d_c \leqslant 0.8$ for metals.

The work on continuum damage theories has been published extensively and applied to both ductile and brittle materials. In this paper we will follow a stress-based continuum damage model proposed by Simo and Ju (1987a,b). It is based on the thermodynamic complementary free potential and results in an additive split of strain vector into the elastic and the inelastic strain parts, whereas the strain-based damage model is based on the free energy potential and results in an additive split of stress vector into initial and inelastic stress parts. The former approach is computationally very efficient and easy to implement in programs which are strain driven.

The damage criterion can be postulated as

$$g({}^t\bar{\tau},{}^t r) = {}^t\bar{\tau}-{}^t r \leqslant 0, \tag{1}$$

where ${}^t\bar{\tau}$ is a norm defined according to the stress–strain state and the damage mechanism assumed for the material and ${}^t r$ is the damage threshold at current time t. In the present work the norm ${}^t\bar{\tau}$ proposed by Oliver *et al.* (1990), which is a norm of the elastic complementary energy with consideration of the difference between compressive damage strength and tensile damage strength, is employed

$${}^t\bar{\tau} = \gamma({}^t\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{D}_{\mathrm{e}}^{-1}\,{}^t\boldsymbol{\sigma})^{1/2}, \tag{2}$$

where $\mathbf{D}_{\mathrm{e}}$ is the elastic modulus matrix, $\boldsymbol{\sigma}$ is the current stress, and γ is given by

$$\gamma = \theta+\frac{1-\theta}{n}, \tag{3}$$

where n is the ratio of the compressive over the tensile damage strength. These damage strengths are the compressive and tensile stresses at which degradation of elastic modulus commences. θ is defined as

$$\theta = \frac{\sum_{i=1}^{3}\langle\sigma^i\rangle}{\sum_{i=1}^{3}|\sigma^i|}. \tag{4}$$

Here $\sigma^i (i = 1, 2, 3)$ are principal undamaged stresses and

$$\langle\sigma^i\rangle = \begin{cases}\sigma^i & \text{if } \sigma^i > 0\\ 0 & \text{otherwise.}\end{cases} \tag{5}$$

Setting the coefficient $\gamma \equiv 1$ expression (2) becomes the norm used in the Simo and Ju (1987a,b) damage model.

If ${}^0 r$ denotes the initial damage threshold of the material, similar to the initial yield strength in the yield criterion, we must have for any current time t that

$${}^t r = \max({}^0 r, {}^{\max}\bar{\tau}), \tag{6}$$

where ${}^{\max}\bar{\tau}$ is the maximum value of $\bar{\tau}$ over the time period from 0 to t.

The damage criterion (1) states that damage in the material is initiated when the elastic complementary energy norm $\bar{\tau}$ exceeds the initial damage threshold ${}^0 r$. Assuming that d is a function of the complementary energy norm $\bar{\tau}$ only,

$$d = d(\bar{\tau}), \tag{7}$$

then its evolution can be defined by a rate equation

$$\dot{d} = \dot{\mu}H(\bar{\tau}, d) = \dot{\mu}\frac{\partial d}{\partial\bar{\tau}}, \tag{8}$$

where $\dot{\mu} \geqslant 0$ is the damage consistency parameter.

According to the damage condition (1), the definitions (2) and (7), we have

$$\dot{\mu} = \frac{\gamma^2}{\bar{\tau}}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{D}_{\mathrm{e}}^{-1}\dot{\boldsymbol{\sigma}}, \tag{9}$$

where the coefficient γ is assumed to be a constant during a load step.

The Hoffman criterion (Hoffman, 1967) is used to describe the yield behaviour for pressure dependent anisotropic nonlinear materials. It contains as special cases, the Hill criterion (Hill, 1947), the modified von Mises criterion (Raghava *et al.*, 1973), and the von Mises criterion. The geometry of the Hoffman yield surface is determined by six uniaxial tensile and compressive yield strengths in three orthogonal axes and three shear yield strengths in three orthogonal planes. The associated theory of plasticity is employed and the form of the flow role in plasticity follows the normality principle. The classical plasticity theories, in which the creep and plastic strains are postulated to arise from two independent constitutive laws, are used in the present coupled creep-elastoplastic-damage analysis; consequently the inelastic strain is assumed to decompose into two separate parts.

As implicit integration is employed to determine a point in the stress space jointly satisfying the creep law and the yield and damage criteria, the incremental creep strain, plastic strain and damage are dependent on the final stress state of the current load step. The governing equations, therefore, are coupled and a Newton iteration is used to simultaneously solve for these increments using a standard backward Euler integration.

In the present paper we will put our emphasis on the integration algorithms for the creep-elastoplastic-damage analysis of anisotropic nonlinear materials with specialization to isotropic materials. After reviewing the governing equations for the analysis in Section 2, a general algorithm for anisotropic materials is derived in Section 3. The stress vector and scalar state variables for the increments in creep, plasticity and damage are taken as the primary unknowns in the Newton–Raphson iteration at a Gauss point and updated simultaneously. In Section 4, Jetteur's (1986) algorithm which is applicable to an isotropic elastoplastic material in a state of plane stress is extended for coupled creep-elastoplastic-damage analysis. In Section 5, we discuss an algorithm for isotropic nonlinear materials described by the von Mises yield criterion, which integrates the three-dimensional solid, plane strain and axisymmetric solid stress models. The algorithm takes a particularly simple form which makes it a very efficient solution procedure for a large class of problems. In Section 6, a consistent tangent matrix with consideration of the fully coupled effects is derived to ensure quadratic convergence of the global Newton iterative procedure. A coupled analysis procedure with three hierarchical phases to update the stress vector and state variables at a Gauss point is briefly described in Section 7. Some numerical results are given to illustrate the performance of the present algorithms in Section 8.

2. GOVERNING EQUATIONS OF COUPLED CREEP-ELASTOPLASTIC-DAMAGE ANALYSIS

The total strain $\boldsymbol{\varepsilon}$ is decomposed into elastic strain $\boldsymbol{\varepsilon}^{e}$ and inelastic strain $\boldsymbol{\varepsilon}^{i}$. In addition, inelastic strain $\boldsymbol{\varepsilon}^{i}$ is additively decomposed into plastic strain $\boldsymbol{\varepsilon}^{p}$ and creep strain $\boldsymbol{\varepsilon}^{c}$:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{e}+\boldsymbol{\varepsilon}^{i} = \boldsymbol{\varepsilon}^{e}+\boldsymbol{\varepsilon}^{p}+\boldsymbol{\varepsilon}^{c}. \tag{10}$$

Based on the thermodynamics of an irreversible process, the elastic strain vector for the material with linear elastic behaviour is given by

$$\boldsymbol{\varepsilon}^{e} = d_{\sigma}\mathbf{D}_{e}^{-1}\boldsymbol{\sigma} \tag{11}$$

with

$$d_{\sigma} = \frac{1}{1-d}, \tag{12}$$

then the total stress can be written as

$$\boldsymbol{\sigma} = (1-d)\mathbf{D}_{e}\boldsymbol{\varepsilon}^{e} = (1-d)\mathbf{D}_{e}(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{i}). \tag{13}$$

The Hoffman yield surface can be written in matrix-vector notation as

$$F = \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{P}\boldsymbol{\sigma} + \mathbf{p}^{\mathrm{T}}\boldsymbol{\sigma} - \tfrac{1}{3}\kappa^2 = 0. \tag{14}$$

For the case of a three-dimensional solid, the stress vector takes the form

$$\boldsymbol{\sigma} = (\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{12} \quad \sigma_{23} \quad \sigma_{31})$$

and the corresponding stress potential matrix $\mathbf{P}$ and vector $\mathbf{p}$ are

$$\mathbf{P} = \begin{bmatrix} 1/3(\alpha_{12}+\alpha_{13}) & -1/3\alpha_{12} & -1/3\alpha_{13} & & & \\ -1/3\alpha_{12} & 1/3(\alpha_{23}+\alpha_{12}) & -1/3\alpha_{23} & & & \\ -1/3\alpha_{13} & -1/3\alpha_{23} & 1/3(\alpha_{13}+\alpha_{23}) & & & \\ & & & 2\alpha_{44} & & \\ & & & & 2\alpha_{55} & \\ & & & & & 2\alpha_{66} \end{bmatrix}, \tag{15}$$

$$\mathbf{p} = (\alpha_{11} \quad \alpha_{22} \quad \alpha_{33} \quad 0 \quad 0 \quad 0), \tag{16}$$

where

$$\alpha_{12} = \left(\frac{1}{\sigma^{\mathrm{t}}_{11,y}\sigma^{\mathrm{c}}_{11,y}} + \frac{1}{\sigma^{\mathrm{t}}_{22,y}\sigma^{\mathrm{c}}_{22,y}} - \frac{1}{\sigma^{\mathrm{t}}_{33,y}\sigma^{\mathrm{c}}_{33,y}}\right)\kappa^2,$$

$$\alpha_{23} = \left(\frac{1}{\sigma^{\mathrm{t}}_{22,y}\sigma^{\mathrm{c}}_{22,y}} + \frac{1}{\sigma^{\mathrm{t}}_{33,y}\sigma^{\mathrm{c}}_{33,y}} - \frac{1}{\sigma^{\mathrm{t}}_{11,y}\sigma^{\mathrm{c}}_{11,y}}\right)\kappa^2,$$

$$\alpha_{13} = \left(\frac{1}{\sigma^{\mathrm{t}}_{33,y}\sigma^{\mathrm{c}}_{33,y}} + \frac{1}{\sigma^{\mathrm{t}}_{11,y}\sigma^{\mathrm{c}}_{11,y}} - \frac{1}{\sigma^{\mathrm{t}}_{22,y}\sigma^{\mathrm{c}}_{22,y}}\right)\kappa^2,$$

$$\alpha_{44} = \frac{\kappa^2}{3\sigma^2_{12,y}}, \quad \alpha_{55} = \frac{\kappa^2}{3\sigma^2_{23,y}}, \quad \alpha_{66} = \frac{\kappa^2}{3\sigma^2_{31,y}}, \tag{17}$$

$$\alpha_{ii} = \frac{\sigma^{\mathrm{c}}_{ii,y} - \sigma^{\mathrm{t}}_{ii,y}}{3\sigma^{\mathrm{c}}_{ii,y}\sigma^{\mathrm{t}}_{ii,y}}\kappa^2 \quad (1 \leqslant i \leqslant 3), \tag{18}$$

$\sigma_{ij}(i, j = 1, 2, 3)$ are uniaxial normal and shear yield stresses, and where superscripts t and c stand for tension and compression, respectively. For pressure dependent materials κ^2 is defined as

$$\kappa^2 = \sigma^{\mathrm{c}}_y\sigma^{\mathrm{t}}_y, \tag{19}$$

where σ^{c}_y and σ^{t}_y are generic compressive and tensile yield stresses. They can be assumed to be piecewise linear functions of the effective plastic strain $\bar{\varepsilon}^{\mathrm{p}}$ and take the following forms:

$$\sigma^{\mathrm{t}}_y = \sigma^{\mathrm{t}}_{y,0} + h^{\mathrm{t}}\bar{\varepsilon}^{\mathrm{p}}, \tag{20a}$$

$$\sigma^{\mathrm{c}}_y = \sigma^{\mathrm{c}}_{y,0} + h^{\mathrm{c}}\bar{\varepsilon}^{\mathrm{p}}. \tag{20b}$$

With the use of only one internal state variable $\bar{\varepsilon}^{\mathrm{p}}$ to the anisotropic yield criterion of Hoffman, a proportional hardening formulation is assumed in the following analysis, in which

$$\frac{h_{ii}^{t}(\bar{\varepsilon}^{p})}{\sigma_{ii,y}^{t}} = C^{t} = \frac{h^{t}(\bar{\varepsilon}^{p})}{\sigma_{y}^{t}},$$

$$\frac{h_{ii}^{c}(\bar{\varepsilon}^{p})}{\sigma_{ii,y}^{c}} = C^{c} = \frac{h^{c}(\bar{\varepsilon}^{p})}{\sigma_{y}^{c}} \quad (i = 1,2,3),$$

$$\frac{h_{12}(\bar{\varepsilon}^{p})}{\sigma_{12,y}} = \frac{h_{23}(\bar{\varepsilon}^{p})}{\sigma_{23,y}} = \frac{h_{31}(\bar{\varepsilon}^{p})}{\sigma_{31,y}} = \tfrac{1}{2}(C^{t}+C^{c}). \tag{20c}$$

Consequently, the coefficients of matrix **P** remain constant whilst the pressure-dependent vector **p** is a simple function of $\bar{\varepsilon}^{p}$. The considerations for the application of strain hardening with the Hoffman yield criterion will be discussed in detail in another paper.

The introduction of strain hardening in this form enforces the original yield surface projected on the π plane to retain its shape as it expands during the strain hardening. Additionally, the intersection of the yield surface with the hydrostatic axis translates according to the relative hardening rates (20).

The accumulated effective plastic strain $\bar{\varepsilon}^{p}$ can be calculated by the integration of incremental effective plastic strain $\Delta\bar{\varepsilon}^{p}$ as

$$\bar{\varepsilon}^{p} = \int \Delta\bar{\varepsilon}^{p}, \tag{21}$$

where the incremental effective plastic strain $\Delta\bar{\varepsilon}^{p}$ is defined as

$$\Delta\bar{\varepsilon}^{p} = (\tfrac{2}{3}(\Delta\boldsymbol{\varepsilon}^{p})^{T}\mathbf{M}\Delta\boldsymbol{\varepsilon}^{p})^{1/2}, \tag{22}$$

where for the full three-dimensional stress state **M** is

$$\mathbf{M} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1/2 & & \\ & & & & 1/2 & \\ & & & & & 1/2 \end{bmatrix} \tag{23}$$

and $\Delta\boldsymbol{\varepsilon}^{p}$ is the corresponding plastic strain vector.

The creep function $\bar{\varepsilon}^{c}$ is assumed to be independent of hydrostatic stress in the form

$$\bar{\varepsilon}^{c} = f(\bar{\sigma}, \Delta\bar{\varepsilon}^{c}, t), \tag{24}$$

where the creep effective stress is defined as

$$\bar{\sigma} = (\tfrac{3}{2}\boldsymbol{\sigma}^{T}\mathbf{P}\boldsymbol{\sigma})^{1/2}. \tag{25}$$

The increment creep condition for the time step can then be defined as

$$\Gamma = \Delta\bar{\varepsilon}^{c} - \frac{\partial\bar{\varepsilon}^{c}}{\partial t}(\bar{\sigma}, \Delta\bar{\varepsilon}^{c})\Delta t = \Delta\bar{\varepsilon}^{c} - \dot{f}(\bar{\sigma}, \Delta\bar{\varepsilon}^{c})\Delta t = 0. \tag{26}$$

It is remarked that as the variation of creep strain with time in the incremental creep condition is calculated it is assumed that the effective stress and the creep strain are constant with respect to time. The differentiation of the incremental creep $\Delta\bar{\varepsilon}^{c}$ with respect to time gives

$$\Delta\dot{\bar{\varepsilon}}^{c} = \left(\frac{\partial \dot{f}}{\partial \bar{\sigma}}\dot{\bar{\sigma}} + \frac{\partial \dot{f}}{\partial(\Delta\bar{\varepsilon}^{c})}\Delta\dot{\bar{\varepsilon}}^{c}\right)\Delta t. \tag{27}$$

It results in

$$\Delta\dot{\bar{\varepsilon}}^{c} = \frac{\dfrac{\partial \dot{f}}{\partial \bar{\sigma}}\Delta t}{1 - \dfrac{\partial \dot{f}}{\partial(\Delta\bar{\varepsilon}^{c})}\Delta t}\dot{\bar{\sigma}}. \tag{28}$$

The calculation of the derivative of $\bar{\sigma}$ with respect to time from (25) and substitution of $\dot{\bar{\sigma}}$ into (28) gives

$$\Delta\dot{\bar{\varepsilon}}^{c} = \beta_{c}\frac{3}{2\bar{\sigma}}\boldsymbol{\sigma}^{T}\mathbf{P}\dot{\boldsymbol{\sigma}}, \tag{29}$$

$$\beta_{c} = \frac{\dfrac{\partial \dot{f}}{\partial \bar{\sigma}}\Delta t}{1 - \dfrac{\partial \dot{f}}{\partial(\Delta\bar{\varepsilon}^{c})}\Delta t}. \tag{30}$$

If plastic yielding occurs simultaneously with creep it is numerically advantageous to express (25) in terms of the yield stress as

$$\bar{\sigma} = (\kappa^{2} - 3\mathbf{p}^{T}\boldsymbol{\sigma})^{1/2}. \tag{31}$$

With the use of the damage consistency condition, the damage consistency parameter μ for an incremental step from time t to $t+\Delta t$ can be written as

$$\begin{aligned} \mu &= {}^{t+\Delta t}r - {}^{t}r = {}^{t+\Delta t}\bar{\tau} - {}^{t}r \\ &= \gamma({}^{t+\Delta t}\boldsymbol{\sigma}\,\mathbf{D}_{e}^{-1}\,{}^{t+\Delta t}\boldsymbol{\sigma})^{1/2} - {}^{t}r \end{aligned} \tag{32}$$

or in the form of the damage condition

$$\beta = \mu - \gamma({}^{t+\Delta t}\boldsymbol{\sigma}\,\mathbf{D}_{e}^{-1}\,{}^{t+\Delta t}\boldsymbol{\sigma})^{1/2} + {}^{t}r = 0. \tag{33}$$

To consider an incremental step from time t to time $t+\Delta t$, the plastic strain vector ${}^{t+\Delta t}\boldsymbol{\varepsilon}^{p}$ and the effective plastic strain ${}^{t+\Delta t}\bar{\varepsilon}^{p}$ at time $t+\Delta t$ are then determined by integration of the flow and hardening rules over the time step from time t to time $t+\Delta t$:

$${}^{t+\Delta t}\boldsymbol{\varepsilon}^{p} = {}^{t}\boldsymbol{\varepsilon}^{p} + \lambda({}^{t+\Delta t}\mathbf{P}\,{}^{t+\Delta t}\boldsymbol{\sigma} + {}^{t+\Delta t}\mathbf{p}), \tag{34}$$

$${}^{t+\Delta t}\bar{\varepsilon}^{p} = {}^{t}\bar{\varepsilon}^{p} + \lambda[\tfrac{2}{3}({}^{t+\Delta t}\mathbf{P}\,{}^{t+\Delta t}\boldsymbol{\sigma} + {}^{t+\Delta t}\mathbf{p})^{T}\mathbf{M}({}^{t+\Delta t}\mathbf{P}\,{}^{t+\Delta t}\boldsymbol{\sigma} + {}^{t+\Delta t}\mathbf{p})]^{1/2}. \tag{35}$$

Similarly the creep strain ${}^{t+\Delta t}\boldsymbol{\varepsilon}^{c}$ is determined as

$${}^{t+\Delta t}\boldsymbol{\varepsilon}^{c} = {}^{t}\boldsymbol{\varepsilon}^{c} + \frac{3}{2\bar{\sigma}}\Delta\bar{\varepsilon}^{c}\,{}^{t+\Delta t}\mathbf{P}\,{}^{t+\Delta t}\boldsymbol{\sigma}, \tag{36}$$

where the second terms on the right-hand sides of eqns (34) and (36) stand for the incremental plastic and creep strain vectors, $\Delta\boldsymbol{\varepsilon}^{p}$ and $\Delta\boldsymbol{\varepsilon}^{c}$, for the backward Euler algorithm.

Substitution of eqns (34) and (36) into (13) at time $t+\Delta t$ gives

$$\mathbf{A}\,^{t+\Delta t}\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{E}} - (1-{}^{t+\Delta t}d)\lambda \mathbf{D}_{\mathrm{e}}^{t+\Delta t}\mathbf{p}, \tag{37}$$

where the algorithmic modulus matrix is defined as

$$\mathbf{A} = \mathbf{I} + (1-{}^{t+\Delta t}d)\left(\lambda + \frac{3}{2\bar{\sigma}}\Delta\bar{\varepsilon}^{\mathrm{c}}\right)\mathbf{D}_{\mathrm{e}}^{t+\Delta t}\mathbf{P} \tag{38}$$

and $\mathbf{I}$ is a unit matrix of dimension equal to that of the number of stress components. The estimation of elastic stress for time $t+\Delta t$ is

$$\boldsymbol{\sigma}^{\mathrm{E}} = (1-{}^{t+\Delta t}d)\mathbf{D}_{\mathrm{e}}({}^{t+\Delta t}\boldsymbol{\varepsilon} - {}^{t}\boldsymbol{\varepsilon}^{i}). \tag{39}$$

The governing equations for the local updates at a Gauss point can then be summarized as:

$$\mathbf{S} = \mathbf{A}\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{E}} + (1-d)\lambda\mathbf{D}_{\mathrm{e}}\mathbf{p} = \mathbf{0}, \tag{40}$$

$$F^{a} = \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{P}\boldsymbol{\sigma} + \mathbf{p}^{\mathrm{T}}\boldsymbol{\sigma} - \tfrac{1}{3}\kappa^{2} = 0 \quad \text{(for elastoplasticity)} \tag{41a}$$

or

$$F^{b} = \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{P}\boldsymbol{\sigma} - \tfrac{1}{3}\bar{\sigma}^{2} = 0, \quad \text{(for elastic-creep)} \tag{41b}$$

$$\Gamma = \Delta\bar{\varepsilon}^{\mathrm{c}} - \Delta t \hat{f}(\bar{\sigma}, t+\Delta t, \Delta\bar{\varepsilon}^{\mathrm{c}}) = 0, \tag{42}$$

$$\beta = \mu - \gamma(\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{D}_{\mathrm{e}}^{-1}\boldsymbol{\sigma})^{1/2} + r = 0. \tag{43}$$

It is observed that the number of governing equations above is $n_\sigma + 3$, where n_σ is the number of stress components. The primary unknowns to be solved for the local updates are: the stress vector $\boldsymbol{\sigma}$ with n_σ components, λ (or $\bar{\sigma}$ for elastic-creep); $\Delta\bar{\varepsilon}^{\mathrm{c}}$ and μ.

3. A GENERAL INTEGRATION ALGORITHM FOR ANISOTROPIC MATERIALS WITH THE HOFFMAN CRITERION

As the governing eqns (40)–(43) are simultaneously fulfilled at the current iteration k:

$$\mathbf{S}_{k} = \mathbf{S}_{k-1} + \mathrm{d}\mathbf{S} = 0, \tag{44}$$

$$F^{a}_{k} = F^{a}_{k-1} + \mathrm{d}F^{a} = 0, \quad (\text{or } F^{b}_{k} = F^{b}_{k-1} + \mathrm{d}F^{b} = 0), \tag{45}$$

$$\Gamma_{k} = \Gamma_{k-1} + \mathrm{d}\Gamma = 0, \tag{46}$$

$$\beta_{k} = \beta_{k-1} + \mathrm{d}\beta = 0. \tag{47}$$

The Newton–Raphson iteration for the update of the stress vector, plastic multiplier (or effective stress), effective creep strain and damage consistency parameter at a Gauss point can be given as

$$\begin{bmatrix} \frac{\partial \mathbf{S}}{\partial \boldsymbol{\sigma}} & \frac{\partial \mathbf{S}}{\partial \lambda} & \frac{\partial \mathbf{S}}{\partial \bar{\sigma}} & \frac{\partial \mathbf{S}}{\partial \Delta\bar{\varepsilon}^{c}} & \frac{\partial \mathbf{S}}{\partial \mu} \\ \frac{\partial F^{a}}{\partial \boldsymbol{\sigma}} & \frac{\partial F^{a}}{\partial \lambda} & \frac{\partial F^{a}}{\partial \bar{\sigma}} & \frac{\partial F^{a}}{\partial \Delta\bar{\varepsilon}^{c}} & \frac{\partial F^{a}}{\partial \mu} \\ \frac{\partial F^{b}}{\partial \boldsymbol{\sigma}} & \frac{\partial F^{b}}{\partial \lambda} & \frac{\partial F^{b}}{\partial \bar{\sigma}} & \frac{\partial F^{b}}{\partial \Delta\bar{\varepsilon}^{c}} & \frac{\partial F^{b}}{\partial \mu} \\ \frac{\partial \Gamma}{\partial \boldsymbol{\sigma}} & \frac{\partial \Gamma}{\partial \lambda} & \frac{\partial \Gamma}{\partial \bar{\sigma}} & \frac{\partial \Gamma}{\partial \Delta\bar{\varepsilon}^{c}} & \frac{\partial \Gamma}{\partial \mu} \\ \frac{\partial \beta}{\partial \boldsymbol{\sigma}} & \frac{\partial \beta}{\partial \lambda} & \frac{\partial \beta}{\partial \bar{\sigma}} & \frac{\partial \beta}{\partial \Delta\bar{\varepsilon}^{c}} & \frac{\partial \beta}{\partial \mu} \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{\sigma} \\ \Delta\lambda \\ \Delta\bar{\sigma} \\ \delta\Delta\bar{\varepsilon}^{c} \\ \Delta\mu \end{Bmatrix} = \begin{Bmatrix} -\mathbf{S}_{k-1} \\ -F^{a}_{k-1} \\ -F^{b}_{k-1} \\ -\Gamma_{k-1} \\ -\beta_{k-1} \end{Bmatrix} \tag{48}$$

with the entries in the Jacobian matrix of eqn (48) being

$$\frac{\partial \mathbf{S}}{\partial \boldsymbol{\sigma}} = \mathbf{A} \quad \text{(for elasticity)}, \tag{49a1}$$

$$\frac{\partial \mathbf{S}}{\partial \boldsymbol{\sigma}} = \mathbf{A} - (1-d)\frac{9\Delta\bar{\varepsilon}^{c}}{4\bar{\sigma}^{3}}\mathbf{D}_{e}\mathbf{P}\boldsymbol{\sigma}\boldsymbol{\sigma}^{T}\mathbf{P} + (1-d)\lambda\mathbf{D}_{e}\frac{d\mathbf{p}}{d\bar{\varepsilon}^{p}}\frac{\partial \bar{\varepsilon}^{p^{T}}}{\partial \boldsymbol{\sigma}} \quad \text{(for elastoplasticity)}, \tag{49a2}$$

$$\frac{\partial \mathbf{S}}{\partial \bar{\sigma}} = -(1-d)\frac{3\Delta\bar{\varepsilon}^{c}}{2\bar{\sigma}^{2}}\mathbf{D}_{e}\mathbf{P}\boldsymbol{\sigma} \quad \text{(for elasticity)}, \tag{49b1}$$

$$\frac{\partial \mathbf{S}}{\partial \lambda} = (1-d)\mathbf{D}_{e}\left[\mathbf{P}\boldsymbol{\sigma} + \mathbf{p} + \lambda\frac{d\mathbf{p}}{d\bar{\varepsilon}^{p}}\frac{\partial \bar{\varepsilon}^{p}}{\partial \lambda} - \frac{3\Delta\bar{\varepsilon}^{c}}{4\bar{\sigma}^{3}}\left(2\kappa\frac{\partial \kappa}{\partial \lambda} - 3\boldsymbol{\sigma}^{T}\frac{d\mathbf{p}}{d\bar{\varepsilon}^{p}}\frac{\partial \bar{\varepsilon}^{p}}{\partial \lambda}\right)\mathbf{P}\boldsymbol{\sigma}\right] \quad \text{(for elastoplasticity)}, \tag{49b2}$$

$$\frac{\partial \mathbf{S}}{\partial \Delta\bar{\varepsilon}^{c}} = (1-d)\frac{3}{2\bar{\sigma}}\mathbf{D}_{e}\mathbf{P}\boldsymbol{\sigma}, \tag{49c}$$

$$\frac{\partial \mathbf{S}}{\partial \mu} = H\left[\frac{1}{1-d}\boldsymbol{\sigma}^{E} - \mathbf{D}_{e}\left(\lambda(\mathbf{P}\boldsymbol{\sigma} + \mathbf{p}) + \frac{3}{2\bar{\sigma}}\Delta\bar{\varepsilon}^{c}\mathbf{P}\boldsymbol{\sigma}\right)\right], \tag{49d}$$

$$\frac{\partial F^{a}}{\partial \boldsymbol{\sigma}} = \mathbf{P}\boldsymbol{\sigma} + \mathbf{p} + C_{\varepsilon}\frac{\partial \bar{\varepsilon}^{p}}{\partial \boldsymbol{\sigma}}, \tag{49e}$$

with

$$C_{\varepsilon} = \boldsymbol{\sigma}^{T}\frac{d\mathbf{p}}{d\bar{\varepsilon}^{p}} - \frac{2}{3}\kappa\frac{d\kappa}{d\bar{\varepsilon}^{p}}, \tag{49f}$$

$$\frac{\partial F^{a}}{\partial \lambda} = C_{\varepsilon}\frac{\partial \bar{\varepsilon}^{p}}{\partial \lambda}, \tag{49g}$$

$$\frac{\partial F^{a}}{\partial \Delta\bar{\varepsilon}^{c}} = \frac{\partial F^{a}}{\partial \mu} = \frac{\partial F^{b}}{\partial \Delta\bar{\varepsilon}^{c}} = \frac{\partial F^{b}}{\partial \mu} = 0, \tag{49h}$$

$$\frac{\partial F^{b}}{\partial \boldsymbol{\sigma}} = \mathbf{P}\boldsymbol{\sigma}, \tag{49i}$$

$$\frac{\partial F^{b}}{\partial \bar{\sigma}} = -\frac{2}{3}\bar{\sigma}, \tag{49j}$$

$$\frac{\partial \Gamma}{\partial \boldsymbol{\sigma}} = \mathbf{0} \quad \text{(elasticity)}, \tag{49k1}$$

$$\frac{\partial \Gamma}{\partial \boldsymbol{\sigma}} = \Delta t \frac{\partial \dot{f}}{\partial \bar{\sigma}} \frac{3}{2\bar{\sigma}} \mathbf{p}^{\mathrm{T}} \quad \text{(elastoplasticity)}, \tag{49k2}$$

$$\frac{\partial \Gamma}{\partial \bar{\sigma}} = -\Delta t \frac{\partial \dot{f}}{\partial \bar{\sigma}} \quad \text{(elasticity)}, \tag{49m1}$$

$$\frac{\partial \Gamma}{\partial \lambda} = -\Delta t \frac{\partial \dot{f}}{\partial \bar{\sigma}} \frac{1}{2\bar{\sigma}} \left(2\kappa \frac{\partial \kappa}{\partial \lambda} - 3\boldsymbol{\sigma}^{\mathrm{T}} \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\bar{\varepsilon}^{\mathrm{p}}} \frac{\partial \bar{\varepsilon}^{\mathrm{p}}}{\partial \lambda} \right) \quad \text{(elastoplasticity)}, \tag{49m2}$$

$$\frac{\partial \Gamma}{\partial \Delta \bar{\varepsilon}^{\mathrm{c}}} = 1 - \Delta t \frac{\partial \dot{f}}{\partial \Delta \bar{\varepsilon}^{\mathrm{c}}}, \tag{49n}$$

$$\frac{\partial \Gamma}{\partial \mu} = 0, \tag{49o}$$

$$\frac{\partial \beta}{\partial \boldsymbol{\sigma}} = -\frac{\gamma \mathbf{D}_{\mathrm{e}}^{-1} \boldsymbol{\sigma}}{(\boldsymbol{\sigma}^{\mathrm{T}} \mathbf{D}_{\mathrm{e}}^{-1} \boldsymbol{\sigma})^{1/2}} = -\frac{\gamma^2 \mathbf{D}_{\mathrm{e}}^{-1} \boldsymbol{\sigma}}{\bar{\tau}}, \tag{49p}$$

$$\frac{\partial \beta}{\partial \lambda} = \frac{\partial \beta}{\partial \Delta \bar{\varepsilon}^{\mathrm{c}}} = \frac{\partial \beta}{\partial \bar{\sigma}} = 0, \tag{49q}$$

$$\frac{\partial \beta}{\partial \mu} = 1. \tag{49r}$$

The derivatives of the effective plastic strain are calculated from its definition as

$$\frac{\partial \bar{\varepsilon}^{\mathrm{p}}}{\partial \boldsymbol{\sigma}} = \frac{\mathbf{PM}(\mathbf{P}\boldsymbol{\sigma} + \mathbf{p})}{\dfrac{3\bar{a}}{2\lambda} - \dfrac{\mathrm{d}\mathbf{p}^{\mathrm{T}}}{\mathrm{d}\bar{\varepsilon}^{\mathrm{p}}} \mathbf{M}(\mathbf{P}\boldsymbol{\sigma} + \mathbf{p})}, \tag{49s}$$

$$\frac{\partial \bar{\varepsilon}^{\mathrm{p}}}{\partial \lambda} = \frac{\dfrac{3\bar{a}^2}{2\lambda}}{\dfrac{3\bar{a}}{2\lambda} - \dfrac{\mathrm{d}\mathbf{p}^{\mathrm{T}}}{\mathrm{d}\bar{\varepsilon}^{\mathrm{p}}} \mathbf{M}(\mathbf{P}\boldsymbol{\sigma} + \mathbf{p})}, \tag{49t}$$

where

$$\bar{a} = [\tfrac{2}{3}(\mathbf{P}\boldsymbol{\sigma} + \mathbf{p})^{\mathrm{T}} \mathbf{M}(\mathbf{P}\boldsymbol{\sigma} + \mathbf{p})]^{1/2}. \tag{49u}$$

The three state variables λ, $\Delta\bar{\varepsilon}^{\mathrm{c}}$ and μ will be obtained by the accumulation of increments $\Delta\lambda$, $\delta(\Delta\bar{\varepsilon}^{\mathrm{c}})$ and $\Delta\mu$ in the local Newton–Raphson iterations at each Gauss point.

4. ALGORITHM FOR VON MISES POTENTIAL WITH PLANE STRESS

Following Jetteur's (1986) approach, the current stresses for creep-elastoplasticity with consideration of damage in plane stress can be given as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{Bmatrix} = \begin{Bmatrix} \sigma_x^{\mathrm{E}} \\ \sigma_y^{\mathrm{E}} \\ \tau^{\mathrm{E}} \end{Bmatrix} - \frac{1-d}{2\bar{\sigma}} (\lambda + \Delta\bar{\varepsilon}^{\mathrm{c}}) \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} 2\sigma_x - \sigma_y \\ 2\sigma_y - \sigma_x \\ 6\tau \end{Bmatrix} \tag{50}$$

and the definitions for the creep increment and plastic strain increments are

$$\Delta\boldsymbol{\varepsilon}^{c} = \frac{3\Delta\bar{\varepsilon}^{c}}{2\bar{\sigma}}\mathbf{P}\boldsymbol{\sigma}, \tag{51}$$

$$\Delta\boldsymbol{\varepsilon}^{p} = \frac{3\lambda}{2\bar{\sigma}}\mathbf{P}\boldsymbol{\sigma}. \tag{52}$$

Equation (50) can also be written as

$$\sigma_s = \sigma_x + \sigma_y = (\sigma_x^E + \sigma_y^E) \Big/ \left[1 + \frac{1-d}{2\bar{\sigma}}(\lambda + \Delta\bar{\varepsilon}^{c})\frac{E}{1-\nu}\right], \tag{53}$$

$$\sigma_d = \sigma_x - \sigma_y = (\sigma_x^E - \sigma_y^E) \Big/ \left[1 + \frac{1-d}{2\bar{\sigma}}(\lambda + \Delta\bar{\varepsilon}^{c})\frac{3E}{1+\nu}\right], \tag{54}$$

$$\tau = \tau^E \Big/ \left[1 + \frac{1-d}{2\bar{\sigma}}(\lambda + \Delta\bar{\varepsilon}^{c})\frac{3E}{1+\nu}\right], \tag{55}$$

and the estimated stresses, assuming a totally elastic increment, are

$$\boldsymbol{\sigma}^E = \begin{Bmatrix} \sigma_x^E \\ \sigma_y^E \\ \tau^E \end{Bmatrix} = (1-d)\mathbf{D}_e({}^{t+\Delta t}\boldsymbol{\varepsilon} - {}^{t}\boldsymbol{\varepsilon}^{i}), \tag{56}$$

with ${}^{t+\Delta t}\boldsymbol{\varepsilon}$ the total strain at time $t+\Delta t$ and ${}^{t}\boldsymbol{\varepsilon}^{i}$ the inelastic strain at time t. The yield criterion for the plane stress can be written as

$$F = \sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau^2 - \kappa^2(\bar{\varepsilon}^{p}) = 0. \tag{57}$$

Eliminating σ_x, σ_y and τ from (57) using (53)–(55) we obtain a nonlinear equation in λ, $\Delta\bar{\varepsilon}^{c}$ and d:

$$F = \frac{1}{4}\left[\frac{(\sigma_x^E + \sigma_y^E)}{1 + \dfrac{(1-d)(\lambda+\Delta\bar{\varepsilon}^{c})E}{2\bar{\sigma}(1-\nu)}}\right]^2 + \frac{3}{4}\frac{(\sigma_x^E - \sigma_y^E)^2 + 4\tau^{E^2}}{\left[1 + \dfrac{3(1-d)(\lambda+\Delta\bar{\varepsilon}^{c})E}{2\bar{\sigma}(1+\nu)}\right]^2} - \kappa^2 = 0. \tag{58}$$

For an elastic-creep damage analysis without any plastic yielding, κ in eqn (58) is replaced by the effective stress $\bar{\sigma}$. The equation then relates the damage and creep increments to the effective stress.

Eliminating the stress vector explicitly from the damage condition (32) results in a nonlinear equation in λ, $\Delta\bar{\varepsilon}^{c}$ and d.

For elastoplastic-strain dependent creep with damage, the Newton–Raphson iteration for the local updates at a Gauss point can be written as

$$\begin{bmatrix} \partial F/\partial\lambda & \partial F/\partial\Delta\bar{\varepsilon}^{c} & \partial F/\partial\mu \\ \partial\Gamma/\partial\lambda & \partial\Gamma/\partial\Delta\bar{\varepsilon}^{c} & \partial\Gamma/\partial\mu \\ \partial\beta/\partial\lambda & \partial\beta/\partial\Delta\bar{\varepsilon}^{c} & \partial\beta/\partial\mu \end{bmatrix} \begin{Bmatrix} \Delta\lambda \\ \delta\Delta\bar{\varepsilon}^{c} \\ \Delta\mu \end{Bmatrix} = \begin{Bmatrix} -F_{k-1} \\ -\Gamma_{k-1} \\ -\beta_{k-1} \end{Bmatrix}, \tag{59}$$

where the Jacobian of eqn (59) can be explicitly given as

$$\frac{\partial F}{\partial \lambda} = -\frac{1-d}{2\bar{\sigma}} C_3 \left[\frac{(\sigma_x^E + \sigma_y^E)^2 E}{2A^3(1-\nu)} + \frac{3[(\sigma_x^E - \sigma_y^E)^2 + 4\tau^{E^2}]}{2B^3} \frac{3E}{1+\nu} \right] - 2\kappa h, \tag{60a}$$

$$\frac{\partial F}{\partial \Delta \bar{\varepsilon}^c} = -\frac{1-d}{2\bar{\sigma}} \left[\frac{(\sigma_x^E + \sigma_y^E)^2 E}{2A^3(1-\nu)} + \frac{3[(\sigma_x^E - \sigma_y^E)^2 + 4\tau^{E^2}]}{2B^3} \frac{3E}{1+\nu} \right], \tag{60b}$$

$$\frac{\partial F}{\partial \mu} = \begin{Bmatrix} 2\sigma_x - \sigma_y \\ 2\sigma_y - \sigma_x \\ 6\tau \end{Bmatrix}^{\mathrm{T}} \frac{\partial \boldsymbol{\sigma}}{\partial \mu}, \tag{60c}$$

where the components of $\partial \boldsymbol{\sigma} / \partial \mu$ are

$$\frac{\partial \sigma_x}{\partial \mu} = \frac{1}{2}\left(\frac{\partial \sigma_s}{\partial \mu} + \frac{\partial \sigma_d}{\partial \mu} \right); \quad \frac{\partial \sigma_y}{\partial \mu} = \frac{1}{2}\left(\frac{\partial \sigma_s}{\partial \mu} - \frac{\partial \sigma_d}{\partial \mu} \right), \tag{60d}$$

$$\frac{\partial \tau}{\partial \mu} = \frac{-H\left[\frac{1}{1-d} B - \frac{3E}{2\bar{\sigma}(1+\nu)} (\lambda + \Delta \bar{\varepsilon}^c) \right]}{B} \tau, \tag{60e}$$

where

$$\frac{\partial \sigma_s}{\partial \mu} = \frac{-H\left[\frac{1}{1-d} A - \frac{E}{2\bar{\sigma}(1+\nu)} (\lambda + \Delta \bar{\varepsilon}^c) \right]}{A} \sigma_s, \tag{60f}$$

$$\frac{\partial \sigma_d}{\partial \mu} = \frac{-H\left[\frac{1}{1-d} B - \frac{3E}{2\bar{\sigma}(1+\nu)} (\lambda + \Delta \bar{\varepsilon}^c) \right]}{B} \sigma_d. \tag{60g}$$

Further components of (59) are

$$\frac{\partial \Gamma}{\partial \lambda} = -\Delta t \frac{\partial \dot{f}}{\partial \bar{\sigma}} h, \tag{60h}$$

$$\frac{\partial \Gamma}{\partial \Delta \bar{\varepsilon}^c} = 1 - \Delta t \frac{\partial \dot{f}}{\partial \Delta \bar{\varepsilon}^c}, \tag{60i}$$

$$\frac{\partial \Gamma}{\partial \mu} = 0, \tag{60j}$$

and for the final row of (59)

$$\frac{\partial \beta}{\partial \lambda} = -\frac{\gamma^2}{\bar{\tau}} (\mathbf{D}_e^{-1} \boldsymbol{\sigma})^{\mathrm{T}} \frac{\partial \boldsymbol{\sigma}}{\partial \lambda}, \tag{60k}$$

$$\frac{\partial \beta}{\partial \Delta \bar{\varepsilon}^c} = -\frac{\gamma^2}{\bar{\tau}} (\mathbf{D}_e^{-1} \boldsymbol{\sigma})^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial \Delta \bar{\varepsilon}^c}, \tag{60l}$$

$$\frac{\partial \beta}{\partial \mu} = 1 - \frac{\gamma^2}{\bar{\tau}} (\mathbf{D}_e^{-1} \boldsymbol{\sigma})^{\mathrm{T}} \frac{\partial \boldsymbol{\sigma}}{\partial \mu}, \tag{60m}$$

with

$$\frac{\partial \sigma_s}{\partial \lambda} = -\frac{(1-d)EC_3\sigma_s}{2\bar{\sigma}(1-\nu)A}, \quad \frac{\partial \sigma_d}{\partial \lambda} = -\frac{3(1-d)EC_3\sigma_d}{2\bar{\sigma}(1+\nu)B}, \quad \frac{\partial \tau}{\partial \lambda} = -\frac{3(1-d)EC_3\tau}{2\bar{\sigma}(1+\nu)B}, \tag{60n}$$

$$\frac{\partial \sigma_s}{\partial \Delta \bar{\varepsilon}^c} = -\frac{(1-d)E\sigma_s}{2\bar{\sigma}(1-\nu)A}, \quad \frac{\partial \sigma_d}{\partial \Delta \bar{\varepsilon}^c} = -\frac{3(1-d)E\sigma_d}{2\bar{\sigma}(1+\nu)B}, \quad \frac{\partial \tau}{\partial \Delta \bar{\varepsilon}^c} = -\frac{3(1-d)E\tau}{2\bar{\sigma}(1+\nu)B}. \tag{60o}$$

Here the coefficients A, B and C_3 are given as

$$A = 1+\frac{(1-d)(\lambda+\Delta\bar{\varepsilon}^c)E}{2\bar{\sigma}(1-\nu)}, \tag{60p}$$

$$B = 1+\frac{3(1-d)(\lambda+\Delta\bar{\varepsilon}^c)E}{2\bar{\sigma}(1+\nu)}, \tag{60q}$$

$$C_3 = 1-\frac{h}{\bar{\sigma}}(\lambda+\Delta\bar{\varepsilon}^c). \tag{60r}$$

5. ALGORITHM FOR VON MISES POTENTIAL WITH PLANE STRAIN, AXISYMMETRIC SOLID AND THREE-DIMENSIONAL SOLID

It is observed that for the von Mises stress potential with plane strain, axisymmetric solid and three-dimensional solid, the plastic flow vector calculated at the estimated stress position is identical to that calculated using the stress satisfying the yield criterion. This property allows direct scaling of the estimated stresses to return them to yield surface and has resulted in the development of the radial return algorithm (Simo and Taylor, 1985).

To evaluate the damage condition, a norm of current elastic complementary energy relating to the total stress vector needs to be calculated. To do this we may decompose the stress vector into the two parts; one deviatoric the other hydrostatic:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_d+\boldsymbol{\sigma}_m, \tag{61}$$

where

$$\boldsymbol{\sigma}_d = \mathbf{P}\boldsymbol{\sigma}, \tag{62}$$

$$\boldsymbol{\sigma}_m = 3^{1/2}\sigma_m\mathbf{m}^0, \tag{63}$$

where σ_m is the spherical stress:

$$\sigma_m = (\sigma_{xx}+\sigma_{yy}+\sigma_{zz})/3, \tag{64}$$

$\mathbf{m}^0$ is the unit vector defining the hydrostatic axis, the plane normal to which is known as the π plane, in the stress space:

$$\mathbf{m}^0 = (1/3^{1/2} \quad 1/3^{1/2} \quad 1/3^{1/2} \quad 0 \quad 0 \quad 0)^{\mathrm{T}}. \tag{65}$$

The unit vector of σ_d is evaluated as

$$\boldsymbol{\sigma}_d^0 = \frac{\mathbf{P}\boldsymbol{\sigma}}{\|\mathbf{P}\boldsymbol{\sigma}\|} = \left(\frac{2}{3}\right)^{1/2}\frac{\mathbf{n}}{c}, \tag{66}$$

with the flow vector at the stress point $\boldsymbol{\sigma}$

$$\mathbf{n} = \frac{3}{2\bar{\sigma}}\mathbf{P}\boldsymbol{\sigma}, \tag{67}$$

and a coefficient

$$c = \left(1+\frac{3\boldsymbol{\sigma}_s^{\mathrm{T}}\boldsymbol{\sigma}_s}{\bar{\sigma}^2}\right)^{1/2}, \tag{68}$$

where $\boldsymbol{\sigma}_s$ is a vector defining the shear stresses.

Equation (61) can then be decomposed into its deviatoric and hydrostatic components as

$$\boldsymbol{\sigma} = (\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma}_d^0)\boldsymbol{\sigma}_d^0 + (\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{m}^0)\mathbf{m}^0. \tag{69}$$

Substitution of eqns (66), (67) and (65) into eqn (69) leads to

$$\boldsymbol{\sigma} = \frac{2\bar{\sigma}}{3c^2}\mathbf{n} + \sigma_m\mathbf{m}, \tag{70}$$

where

$$\mathbf{m} = (1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0)^{\mathrm{T}}. \tag{71}$$

As the creep strain vector and the plastic strain vector are defined by (51) and (52), the returned stresses for the von Mises criterion can be written as

$$\boldsymbol{\sigma} = (1-d)(\boldsymbol{\sigma}^{EO} - (\lambda + \Delta\bar{\varepsilon}^c)\mathbf{D}_e\mathbf{n}) \tag{72}$$

with an elastic estimation of the stress, without consideration of damage, being

$$\boldsymbol{\sigma}^{EO} = \mathbf{D}_e({}^{t+\Delta t}\boldsymbol{\varepsilon} - {}^{t}\boldsymbol{\varepsilon}^{i}) \tag{73}$$

and the flow vector at the point $\boldsymbol{\sigma}^{EO}$ in the stress space being

$$\mathbf{n}^E = \frac{3}{2\bar{\sigma}^{EO}}\mathbf{P}\boldsymbol{\sigma}^{EO}, \tag{74}$$

where

$$\bar{\boldsymbol{\sigma}}^{EO} = (\tfrac{3}{2}\boldsymbol{\sigma}^{EO^{\mathrm{T}}}\mathbf{P}\boldsymbol{\sigma}^{EO})^{1/2}. \tag{75}$$

It can be verified that

$$\mathbf{n} = \mathbf{n}^{\mathrm{E}} \tag{76}$$

and then projecting (72) onto the π plane gives

$$\bar{\sigma} = (1-d)(\bar{\sigma}^{EO} - K_{\mathrm{E}}(\lambda + \Delta\bar{\varepsilon}^c)) \tag{77}$$

with the effective elastic stiffness

$$K_{\mathrm{E}} = \frac{3E}{2(1+\nu)}. \tag{78}$$

The summation of the first three equations for the normal stresses in eqn (72) gives

$$\sigma_m = \sigma_m^{EO}(1-d) \tag{79}$$

with

$$\sigma_m^{EO} = (\sigma_x^{EO} + \sigma_y^{EO} + \sigma_z^{EO})/3. \tag{80}$$

Substitution of (76) and (79) into (70) gives

$$\boldsymbol{\sigma} = \frac{2\bar{\sigma}}{3c^2}\mathbf{n}^{\mathrm{E}} + \sigma_m^{EO}(1-d)\mathbf{m}. \tag{81}$$

It is remarked that there are only two independent variables in the expression (81) for the stress vector, i.e. $\bar{\sigma}$ and d.

For creep-elastoplastic-damage with strain dependent creep, the governing equations can be written as

$$F = \bar{\sigma} - (1-d)(\bar{\sigma}^{EO} - K_{\mathrm{E}}(\lambda + \Delta\bar{\varepsilon}^{\mathrm{c}})) = 0, \tag{82}$$

$$\beta = \mu - \bar{\tau}(\bar{\sigma}, d) + {}^{t}r = 0, \tag{83}$$

$$\Gamma = \Delta\bar{\varepsilon}^{\mathrm{c}} - \Delta t \dot{f} = 0. \tag{84}$$

The Jacobian of the Newton–Raphson iteration for the local updates for elastoplastic-strain dependent creep with damage are explicitly given as

$$\begin{aligned}
&\frac{\partial F}{\partial \lambda} = h + (1-d)K_{\mathrm{E}}, \quad \frac{\partial F}{\partial \Delta\bar{\varepsilon}^{\mathrm{c}}} = (1-d)K_{\mathrm{E}}, \\
&\frac{\partial F}{\partial \mu} = H(\bar{\sigma}^{EO} - K_{\mathrm{E}}(\lambda + \Delta\bar{\varepsilon}^{\mathrm{c}})), \\
&\frac{\partial \Gamma}{\partial \lambda} = -\frac{\partial \dot{f}}{\partial \bar{\sigma}}\Delta t h, \quad \frac{\partial \Gamma}{\partial \Delta\bar{\varepsilon}^{\mathrm{c}}} = 1 - \frac{\partial \dot{f}}{\partial \Delta\bar{\varepsilon}^{\mathrm{c}}}\Delta t, \quad \frac{\partial \Gamma}{\partial \mu} = 0, \\
&\frac{\partial \beta}{\partial \lambda} = -\frac{\gamma^2(\mathbf{D}_{\mathrm{e}}^{-1}\boldsymbol{\sigma})^{\mathrm{T}}}{\bar{\tau}}\frac{2}{3}\mathbf{n}^{\mathrm{E}} h\left(\frac{1}{c^2} + \frac{6\boldsymbol{\sigma}_s^{\mathrm{T}}\boldsymbol{\sigma}_s}{\bar{\sigma}^2 c^4}\right), \\
&\frac{\partial \beta}{\partial \Delta\bar{\varepsilon}^{\mathrm{c}}} = 0, \quad \frac{\partial \beta}{\partial \mu} = 1 + \gamma^2 \frac{H}{\bar{\tau}}(\mathbf{D}_{\mathrm{e}}^{-1}\boldsymbol{\sigma})^{\mathrm{T}}\sigma_m^{\mathrm{E}}\mathbf{m}.
\end{aligned} \tag{85}$$

6. CONSISTENT CREEP-ELASTOPLASTIC-DAMAGE TANGENT MATRIX

To evaluate the consistent creep-elastoplastic-damage tangent matrix we start with the total strain at time $t + \Delta t$ and decompose it into its components:

$${}^{t+\Delta t}\boldsymbol{\varepsilon} = {}^{t+\Delta t}\boldsymbol{\varepsilon}^{\mathrm{e}} + {}^{t+\Delta t}\boldsymbol{\varepsilon}^{\mathrm{p}} + {}^{t+\Delta t}\boldsymbol{\varepsilon}^{\mathrm{c}}. \tag{86}$$

Differentiation of both sides of (86) with respect to time gives

$${}^{t+\Delta t}\dot{\boldsymbol{\varepsilon}} = {}^{t+\Delta t}\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + {}^{t+\Delta t}\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + {}^{t+\Delta t}\dot{\boldsymbol{\varepsilon}}^{\mathrm{c}}. \tag{87}$$

The derivative of elastic-damage strain with respect to time gives

$${}^{t+\Delta t}\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} = {}^{t+\Delta t}\dot{d}_\sigma \mathbf{D}_{\mathrm{e}}^{-1}\,{}^{t+\Delta t}\boldsymbol{\sigma} + {}^{t+\Delta t}d_\sigma \mathbf{D}_{\mathrm{e}}^{-1}\,{}^{t+\Delta t}\dot{\boldsymbol{\sigma}}. \tag{88}$$

Calculation of the derivative of d_σ defined in (12) with respect to time and substitution of (9) and (8) into (12) give

$${}^{t+\Delta t}\dot{d}_\sigma = {}^{t+\Delta t}H\frac{\gamma^2}{\bar{\tau}}\,{}^{t+\Delta t}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{D}_{\mathrm{e}}^{-1}\,{}^{t+\Delta t}\dot{\boldsymbol{\sigma}}\frac{1}{(1-d)^2}. \tag{89}$$

We then have from (88) and (89) that

$$ {}^{t+\Delta t}\dot{\boldsymbol{\sigma}} = \mathbf{D}_{ed}\, {}^{t+\Delta t}\dot{\boldsymbol{\varepsilon}}^{e} \tag{90} $$

with elastic-damage tangent matrix $\mathbf{D}_{ed}$ being

$$ \mathbf{D}_{ed} = \left[{}^{t+\Delta t}d_{\sigma}\mathbf{D}_e^{-1} + \mathbf{D}_e^{-1}\, {}^{t+\Delta t}\boldsymbol{\sigma}(\mathbf{D}_e^{-1}\, {}^{t+\Delta t}\boldsymbol{\sigma})^{\mathrm{T}} \frac{1}{\bar{\tau}}\, {}^{t+\Delta t}H \frac{\gamma^2}{(1-d)^2} \right]^{-1}. \tag{91} $$

Equation (90) can be expanded by taking the variations of λ, $\Delta\bar{\varepsilon}^{c}$ and $\boldsymbol{\sigma}$ as

$$ \mathrm{d}\boldsymbol{\sigma} = \mathbf{D}_{ed} \left[\mathrm{d}\boldsymbol{\varepsilon} - \mathrm{d}\lambda \left(\mathbf{P}\boldsymbol{\sigma} + \mathbf{p} + c_{\varepsilon}\frac{\partial \bar{\varepsilon}^{p}}{\partial \boldsymbol{\sigma}} + \lambda \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\bar{\varepsilon}^{p}} \frac{\partial \bar{\varepsilon}^{p}}{\partial \lambda} \right) \right. $$
$$ \left. - \left(\left(\lambda + \frac{3\Delta\bar{\varepsilon}^{c}}{2\bar{\sigma}} \right) \mathbf{P} + \frac{9}{4\bar{\sigma}^2} \left(\beta_c - \frac{\Delta\bar{\varepsilon}^{c}}{\bar{\sigma}} \right) \mathbf{P}\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{P} + \lambda \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\bar{\varepsilon}^{p}} \left(\frac{\partial \bar{\varepsilon}^{p}}{\partial \boldsymbol{\sigma}} \right)^{\mathrm{T}} \right) \mathrm{d}\boldsymbol{\sigma} \right]. \tag{92} $$

The consistency condition of the yield condition (41a) can be given as

$$ \mathrm{d}F = \left(\mathbf{P}\boldsymbol{\sigma} + \mathbf{p} + C_{\varepsilon}\frac{\partial \bar{\varepsilon}^{p}}{\partial \boldsymbol{\sigma}} \right)^{\mathrm{T}} \mathrm{d}\boldsymbol{\sigma} + C_{\varepsilon}\frac{\partial \bar{\varepsilon}^{p}}{\partial \lambda}\,\mathrm{d}\lambda = 0. \tag{93} $$

If the material hardens with the plastic strain, $\mathrm{d}\lambda$ can be obtained directly from (93) as

$$ \mathrm{d}\lambda = -\frac{1}{\dfrac{\partial \bar{\varepsilon}^{p}}{\partial \lambda}} \left(\frac{1}{C_{\varepsilon}} (\mathbf{P}\boldsymbol{\sigma} + \mathbf{p})^{\mathrm{T}} + \frac{\partial \bar{\varepsilon}^{p^{\mathrm{T}}}}{\partial \boldsymbol{\sigma}} \right) \mathrm{d}\boldsymbol{\sigma}. \tag{94} $$

Substitution of eqn (94) into (92) results in

$$ \mathrm{d}\boldsymbol{\sigma} = \mathbf{D}_{edpc}\,\mathrm{d}\boldsymbol{\varepsilon}, \tag{95} $$

where the consistent creep-elastoplastic-damage tangent matrix is given as

$$ \mathbf{D}_{edpc} = \left[\mathbf{D}_{ed}^{-1} + \left(\lambda + \frac{3\Delta\bar{\varepsilon}^{c}}{2\bar{\sigma}} \right) \mathbf{P} + \frac{9}{4\bar{\sigma}^2} \left(\beta_c - \frac{\Delta\bar{\varepsilon}^{c}}{\bar{\sigma}} \right) \mathbf{P}\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{P} + \lambda \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\bar{\varepsilon}^{p}} \left(\frac{\partial \bar{\varepsilon}^{p}}{\partial \boldsymbol{\sigma}} \right)^{\mathrm{T}} \right. $$
$$ \left. - \frac{1}{C_{\varepsilon}\dfrac{\partial \bar{\varepsilon}^{p}}{\partial \lambda}} \left(\mathbf{P}\boldsymbol{\sigma} + \mathbf{p} + \mathbf{c}_{\varepsilon}\frac{\partial \bar{\varepsilon}^{p}}{\partial \boldsymbol{\sigma}} + \lambda \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\bar{\varepsilon}^{p}} \frac{\partial \bar{\varepsilon}^{p}}{\partial \lambda} \right) \left(\mathbf{P}\boldsymbol{\sigma} + \mathbf{p} + C_{\varepsilon}\frac{\partial \bar{\varepsilon}^{p}}{\partial \boldsymbol{\sigma}} \right)^{\mathrm{T}} \right]^{-1}. \tag{96} $$

For an elasto-perfectly plastic material, which does not strain harden, the consistency condition (93) simplifies to

$$ (\mathbf{P}\boldsymbol{\sigma} + \mathbf{p})^{\mathrm{T}}\,\mathrm{d}\boldsymbol{\sigma} = 0. \tag{97} $$

Equation (92) then takes the form:

$$ \mathrm{d}\boldsymbol{\sigma} = \mathbf{A}_{tc}^{-1}[\mathrm{d}\boldsymbol{\varepsilon} - (\mathbf{P}\boldsymbol{\sigma} + \mathbf{p})\,\mathrm{d}\lambda], \tag{98a} $$

where

$$ \mathbf{A}_{tc} = \mathbf{D}_{ed}^{-1} + \left(\lambda + \frac{3\Delta\bar{\varepsilon}^{c}}{2\bar{\sigma}} \right) \mathbf{P} + \frac{9}{4\bar{\sigma}^2} \left(\beta_c - \frac{\Delta\bar{\varepsilon}^{c}}{\bar{\sigma}} \right) \mathbf{P}\boldsymbol{\sigma}\boldsymbol{\sigma}^{\mathrm{T}}\mathbf{P}. \tag{98b} $$

Pre-multiplying $(\mathbf{P}\boldsymbol{\sigma}+\mathbf{p})^{\mathrm{T}}$ for both sides of eqn (98a) with the use of eqn (97) gives the variation of λ as

$$\mathrm{d}\lambda = \frac{(\mathbf{P}\boldsymbol{\sigma}+\mathbf{p})^{\mathrm{T}}\mathbf{A}_{\mathrm{tc}}^{-1}\,\mathrm{d}\boldsymbol{\varepsilon}}{(\mathbf{P}\boldsymbol{\sigma}+\mathbf{p})^{\mathrm{T}}\mathbf{A}_{\mathrm{tc}}^{-1}(\mathbf{P}\boldsymbol{\sigma}+\mathbf{p})}. \tag{99}$$

Substitution of eqn (99) for $\mathrm{d}\lambda$ back into eqn (98a) gives the consistent creep-elastoplastic-damage tangent matrix for an elasto-perfectly plastic material as

$$\mathbf{D}_{\mathrm{edpc}} = \mathbf{A}_{\mathrm{tc}}^{-1} - \frac{\mathbf{A}_{\mathrm{tc}}^{-1}(\mathbf{P}\boldsymbol{\sigma}+\mathbf{p})(\mathbf{P}\boldsymbol{\sigma}+\mathbf{p})^{\mathrm{T}}\mathbf{A}_{\mathrm{tc}}^{-1}}{(\mathbf{P}\boldsymbol{\sigma}+\mathbf{p})^{\mathrm{T}}\mathbf{A}_{\mathrm{tc}}^{-1}(\mathbf{P}\boldsymbol{\sigma}+\mathbf{p})}. \tag{100}$$

7. IMPLEMENTATION OF ALGORITHMS—A COUPLED ANALYSIS PROCEDURE WITH THREE HIERARCHICAL PHASES

In preceding sections, three algorithms have been formulated to update the stress vector $\boldsymbol{\sigma}$ and scalar parameters $\bar{\varepsilon}^{\mathrm{p}}$, $\bar{\varepsilon}^{\mathrm{c}}$ and d with consideration of the coupling effects. In order to ascertain which processes were active over a time increment of Δt a three phase solution procedure is adopted. The stress vector and three scalar parameters at time t

$$({}^{t}\boldsymbol{\sigma}\;{}^{t}\bar{\varepsilon}^{\mathrm{c}}\;{}^{t}\bar{\varepsilon}^{\mathrm{p}}\;{}^{t}d) \tag{101}$$

are updated to the stress vector and scalar parameters at time $t+\Delta t$ as

$$({}^{t+\Delta t}\boldsymbol{\sigma}\;{}^{t+\Delta t}\bar{\varepsilon}^{\mathrm{c}}\;{}^{t+\Delta t}\bar{\varepsilon}^{\mathrm{p}}\;{}^{t+\Delta t}d). \tag{102}$$

The three phase solution procedure for an iteration from i to $i+1$ can be described as follows:

(1) in the first phase, creep is assumed to be the only active process, since as long as the material is in a state of stress, creep will be occurring. A corrector obtained from this phase can be written as

$$({}^{t+\Delta t}\boldsymbol{\sigma}^{*(i+1)}\;{}^{t+\Delta t}\bar{\varepsilon}^{\mathrm{c}*(i+1)}\;{}^{t+\Delta t}\bar{\varepsilon}^{\mathrm{p}(i)}\;{}^{t+\Delta t}d^{(i)}); \tag{103}$$

(2) if the stress state ${}^{t+\Delta t}\boldsymbol{\sigma}^{*(i+1)}$ following the creep relaxation is found to have exceeded the plastic yield stress a coupled creep, elasto-plastic analysis is performed. The corrector of phase 2 is obtained as

$$({}^{t+\Delta t}\boldsymbol{\sigma}^{**(i+1)}\;{}^{t+\Delta t}\bar{\varepsilon}^{\mathrm{c}**(i+1)}\;{}^{t+\Delta t}\bar{\varepsilon}^{\mathrm{p}*(i+1)}\;{}^{t+\Delta t}d^{(i)}); \tag{104}$$

(3) finally, the corrector (104) for the second phase is taken as the predictor to evaluate the complementary energy level. If further damage is indicated the solution is re-calculated assuming all three processes are active. It is possible during this stage that the softening due to material damage prevents material yielding. This is indicated by a negative λ on convergence of the Gauss point iteration. In this case the problem must be re-solved assuming damage and creep only. The corrector of this phase is the result for the iteration at time $t+\Delta t$

$$({}^{t+\Delta t}\boldsymbol{\sigma}^{(i+1)}\;{}^{t+\Delta t}\bar{\varepsilon}^{\mathrm{c}(i+1)}\;{}^{t+\Delta t}\bar{\varepsilon}^{\mathrm{p}(i+1)}\;{}^{t+\Delta t}d^{(i+1)}). \tag{105}$$

8. NUMERICAL EXAMPLES

The different algorithms developed in Sections 3, 4 and 5 are tested against simple problems in which a stress history is prescribed and for which simple analytical solutions may be derived. The plane stress, uniaxial and general anisotropic pressure dependent algorithm should produce identical solutions, with the imposition of appropriate boundary conditions.

Figure 1(a) depicts a bar carrying an axial load. It is restrained against axial displacement, but is free to move in the plane orthogonal to the axial direction so that at equilibrium it is in a state of uniaxial stress. It is modelled using solid, plane stress and bar elements.

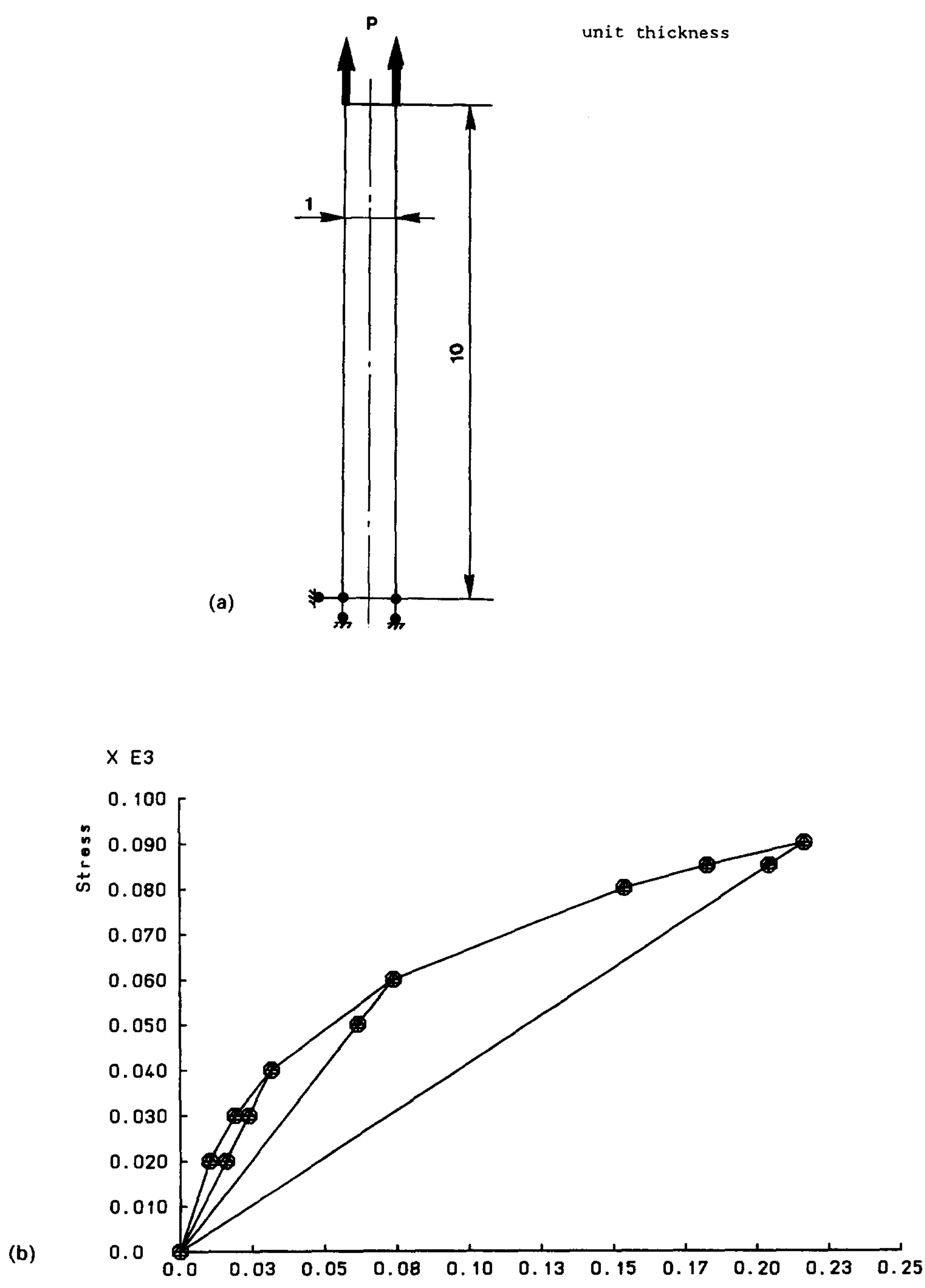


Fig. 1(a). Test bar geometry. (b) Uniaxial elastic-damage test using the three solution algorithms. *Legend*: analytical ——; LUSAS general algorithm (Section 3) ○; LUSAS plane stress algorithm (Section 4) +; LUSAS uniaxial algorithm (Section 5) △.

Example 1. *Elastic damage*

The first example considers elastic damage. The bar is cycled in tension over three cycles with an increasing peak stress:

$$\text{Cycle 1: Stress } \sigma \text{ 0–40, 40–0,}$$
$$\text{Cycle 2: Stress } \sigma \text{ 0–60, 60–0,}$$
$$\text{Cycle 3: Stress } \sigma \text{ 0–90, 90–0.}$$

The damage model of Simo and Ju (1987a, b) defined by the damage potential function

$$d(\bar{\tau}) = 1 - \frac{{}^0r(1-A)}{{}^tr} - A\exp\left[B({}^0r - {}^tr)\right] \tag{106}$$

is used. Setting the elastic and damage parameters as

$$E = 2000, \quad \nu = 0.3,$$
$${}^0r = \tfrac{11}{25}, \quad A = B = 1.0.$$

The axial elastic strain ε is then given by

$$\varepsilon = \frac{\sigma}{(1-d)E} \tag{107}$$

and the damage potential (106) reduces to the form

$$d = 1 - \exp\left(\frac{11}{25} - \frac{1}{E^{1/2}}\sigma\right). \tag{108}$$

The variation of d with stress in (108) is defined by conditions of continuing damage (6). The onset of damage occurs when the initial threshold is breached at a stress level of

$$\sigma = E^{1/2}\tfrac{11}{25}. \tag{109}$$

Figure 1(b) illustrates the limiting stress at increasing levels of strain enforced by (108) as well as the degradation in the elastic stiffness (shown by the decrease in gradient during elastic unloading) following further damage. Exact agreement was achieved using all stress types and algorithms with (108).

Example 2. *Elasto–plastic damage*

Retaining the geometry and material properties of example 1 and defining the plastic yielding as

$$\sigma_y = \sigma_o + h\varepsilon^p, \tag{110}$$

where $\sigma_o = 22.0$ and $h = 100$, the elasto–plastic damage behaviour of the algorithms can be verified.

The bar is subjected to a stress cycle of

$$\text{Phase 1: stress } \sigma \quad 0 \text{ to } 60,$$
$$\text{Phase 2: stress } \sigma \quad 60 \text{ to } -90,$$
$$\text{Phase 3: stress } \sigma \quad -90 \text{ to } 0.$$

Since a stress history is prescribed, the damage and plasticity can be treated as separate processes. Considering damage first; as before it is defined by eqn (108), however, σ is now replaced by its absolute value to account for the stress reversals

$$d = 1 - \exp\left(\frac{11}{25} - \frac{1}{E^{1/2}}|\sigma|\right). \tag{111}$$

Plastic straining occurs once the stress level exceeds the current yield stress σ_y (110). Thus the level of effective plastic strain following yielding is

$$\varepsilon^p = \frac{1}{h}[\sigma_y - \sigma_o], \tag{112}$$

where σ_y is the maximum value of absolute stress in the loading period to date. During the first loading phase, damage is initiated at the stress level defined by (109), followed by elasto–plastic yielding as the yield stress is exceeded. The variation of axial strain for this section of the loading is given by

$$\varepsilon = \frac{\sigma}{\exp\left[\frac{11}{25} - \frac{1}{E^{1/2}}|\sigma|\right]} + \frac{1}{h}[|\sigma| - \sigma_o] \quad (\sigma_o \leqslant \sigma \leqslant 60). \tag{113}$$

Following the stress reversal, phase 2, no further damage or plastic yielding occurs until the axial stress reaches -60, at which point both processes resume. For this stage of the loading, both the effective plastic strain and damage parameters increase. The uniaxial plastic strain component, however, reduces from its peak value. Equation (113) is, therefore, replaced by,

$$\varepsilon = \frac{\sigma}{\exp\left[\frac{11}{25} - \frac{1}{E^{1/2}}|\sigma|\right]} + 2\varepsilon^p_{60} - \frac{1}{h}[|\sigma| - \sigma_o] \quad (-60 > \sigma > -90), \tag{114}$$

where ε^p_{60} is the maximum tensile uniaxial plastic strain attained in the tensile loading phase. During phase 3, elastic unloading takes place. Figure 2(a) shows that exact agreement was obtained.

The present example is re-run by using the Oliver damage model with the ratio $n = 2$ of the compressive over the tensile damage strength. The bar is subjected to a stress cycle of

Phase 1: stress σ 0 to 45,

Phase 2: stress σ 45 to -100,

Phase 3: stress σ -100 to 0.

It is illustrated in Fig. 2(b) that further damage in the load path from $+45$ to -100 is only produced beyond -90 instead of -45 since the model parameter $n = 2$ is specified.

Example 3. *Pressure dependent elasto–plastic damage with creep*

The verification of the solution algorithm for pressure dependent materials follows the same form as the preceding examples with the definition of a stress load path, however, in this case a three-dimensional stress field must be prescribed. For simplicity, the modified von Mises form of the stress potential (14) is adopted. To simplify the calculation of the norm of the elastic complementary energy, Poisson's ratio is set to zero. Young's modulus and the geometry retain the same values as in the preceding examples.

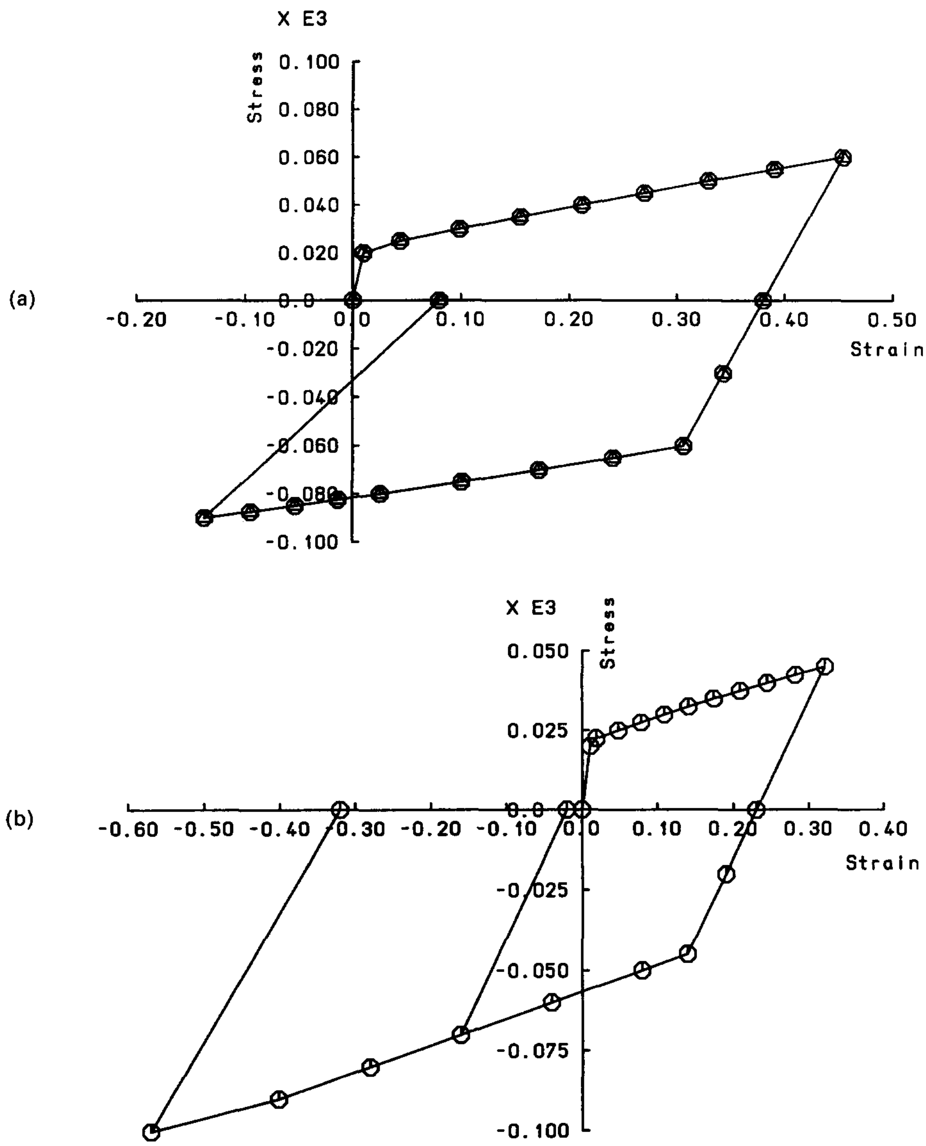


Fig. 2(a). Uniaxial elastic–plastic damage test using plane stress and general solution algorithms. *Legend*: analytical ——; LUSAS general algorithm (Section 3) ⊕; LUSAS plane stress algorithm (Section 4) △. (b) Uniaxial elastoplastic damage test for Oliver damage criterion using three-dimensional solid elements and general solution algorithm. *Legend*: LUSAS general algorithm (Section 3) ⊕.

Splitting the stress vector into its deviatoric and pressure dependent components

$$\sigma = q\mathbf{D}+\alpha\mathbf{H}, \tag{115}$$

where **D** (a unit vector in the π-plane) defines the deviatoric component and **H** (a unit vector along the hydrostatic axis) the hydrostatic. Substituting (115) into (14) yields

$$\tfrac{1}{2}q^2+(\sigma^c-\sigma^t)\frac{\alpha}{3^{1/2}}-\tfrac{1}{3}\sigma^c\sigma^t = 0. \tag{116}$$

Setting the compressive and tensile strain hardening to be

$$\sigma^c = 5+5\varepsilon^p, \quad \sigma^t = 10+10\varepsilon^p \tag{117}$$

and defining the deviatoric and hydrostatic loading parameters as functions of time t

$$q = 4t, \quad \alpha = 2(3^{1/2})t. \tag{118}$$

Substituting (118) into (116), the variation of effective plastic strain with time becomes

$$\varepsilon^{p} = \frac{57^{1/2}-3}{10}t-1, \quad t \geqslant \frac{10}{57^{1/2}-3}. \tag{119}$$

To obtain the individual plastic strain components, it is necessary to specify the deviatoric and hydrostatic load vectors. Considering the three direct stress components only, the vectors are defined as

$$\mathbf{D}^{T} = \left[-\frac{1}{2^{1/2}}\ \frac{1}{2^{1/2}}\ 0\right] \quad \text{and} \quad \mathbf{H}^{T} = \left[\frac{1}{3^{1/2}}\ \frac{1}{3^{1/2}}\ \frac{1}{3^{1/2}}\right]. \tag{120}$$

From (34) the plastic strain components may then be written as

$$\boldsymbol{\varepsilon}^{p} = \lambda[4t\mathbf{D} - 5(3^{-1/2})(1+\varepsilon^{p})\mathbf{H}], \tag{121}$$

substituting the components (121) into (22) gives λ as

$$\lambda = \frac{\varepsilon^{p}}{[\frac{32}{3}t^{2} + \frac{50}{9}(1+\varepsilon^{p})^{2}]^{1/2}}. \tag{122}$$

The plastic strain components can then be evaluated using (120)–(122). In addition to plastic straining, creep strain is also introduced into the problem using the rate form of the power law

$$\dot{\varepsilon}^{c} = A_{c}\bar{\sigma}^{n}t^{m}, \tag{123}$$

where the equivalent stress is given, for the current loading, by

$$\bar{\sigma} = (3/2)^{1/2}q = 4(3/2)^{1/2}t. \tag{124}$$

Introducing the numerical values

$$A_{c} = 0.5 \times 10^{-6}, \quad n = 5, \quad m = -0.5$$

and integrating, the creep strain becomes

$$\varepsilon^{c} = \tfrac{1}{11} \times 10^{-7}[4(3/2)^{1/2}]^{5}t^{11/2} \tag{125}$$

and the component in the x-axis direction is, from (36),

$$\varepsilon_{x}^{c} = -\frac{3^{1/2}}{2}\varepsilon^{c}. \tag{126}$$

Equations (125) and (126) define the variation of creep strain in the x-axis direction with time.

Finally, the damage criterion is evaluated. Substituting (115) into (2) gives

$$\bar{\tau} = \frac{1}{E^{1/2}}[q^{2}+\alpha^{2}]^{1/2} \tag{127}$$

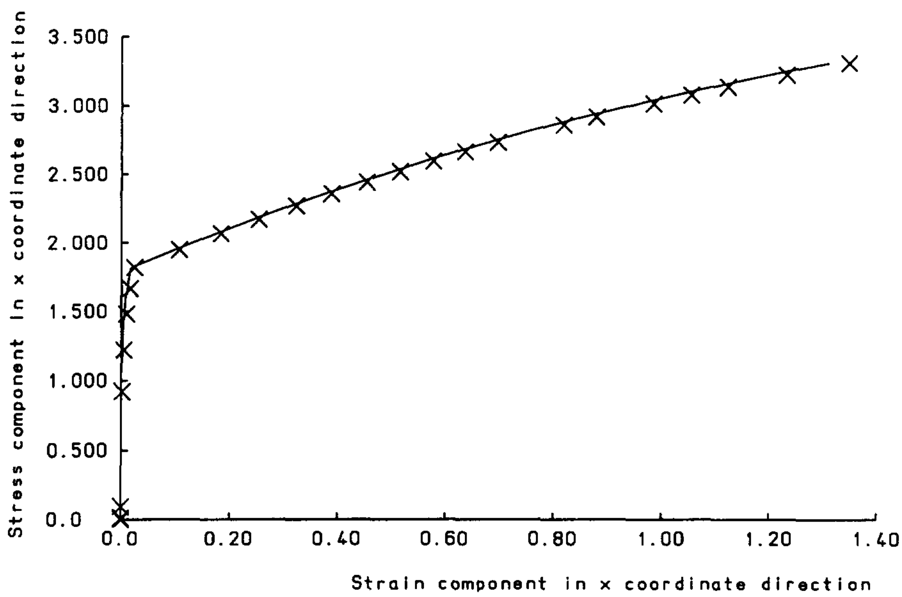


Fig. 3. Prescribed stress loading of pressure dependent material with creep and damage. *Legend*: analytical ——; LUSAS general algorithm (Section 3) ×.

and substituting the loading functions (118) into (127) gives

$$\bar{t} = \left(\frac{28}{E}\right)^{1/2} t. \tag{128}$$

The damage parameters for Simo's model (106) are set to

$$A = B = 1.0, \quad {}^{0}r = 0.08.$$

Thus, once the damage threshold is exceeded, the damage is given by,

$$d = 1 - \exp\left[0.08 - \left(\frac{28}{E}\right)^{1/2} t\right], \quad t > 0.08\left(\frac{E}{28}\right)^{1/2}. \tag{129}$$

Figure 3 illustrates the variation of total strain in the x-direction with the corresponding direct stress component. Again, good agreement is found.

Example 4. *Approximate modelling of a polymer*

The behaviour of a polymer under cyclic loading results in a looping path in which the material stiffness is greater during loading than during unloading. The polymer also exhibits increasing strain as the stress level decreases beyond the point of maximum stress (Browning *et al.*, 1984). This example demonstrates how the form of such a curve may be approximated and how the different components of plasticity, creep, visco-elastic recovery and damage may be combined to reproduce this type of behaviour. Browning *et al.* (1984) model the polymer using a creep and damage function as the rate of loading is seen to have a significant effect on the material behaviour.

The model geometry and uniaxial loading will be as in the first two examples. The cycle of loading is

σ: 0–10–0 in a time of 2.0,

σ: 0–12–0 in a time of 2.4,

σ: 0–14–0 in a time of 2.8,

where the stress varies linearly with time between these points.

As a first approximation to the polymer behaviour, the damage potential is introduced as a linear function:

$$d = \frac{\sigma}{\sigma_{\max}}, \tag{130}$$

where $\sigma_{\max}$ is the maximum permissible stress level. Damage is assumed to start as soon as loading is initiated. Time independent plasticity is also introduced:

$$\sigma = \sigma_o + h\varepsilon^p. \tag{131}$$

The model parameters are chosen to be

$$E = 40, \quad \sigma_{\max} = 20, \quad \sigma_o = 5, \quad h = 50.$$

Figure 4(a) illustrates the resulting time independent stress against strain relationship. Time dependent plasticity is now included in the form

$$\varepsilon^c = A_c \bar{\sigma}^n t^m, \tag{132}$$

where creep parameters are set to

$$A_c = 5 \times 10^{-4}, \quad n = 4, \quad m = 1.$$

As shown in Fig. 4(b), the introduction of creep results in the rounding of the stress–strain curves. In particular, the effect of increasing strain whilst the stress is reducing is now included. This arises from the difference in rates of creep straining to elastic strain recovery. As the stress reduces, the elastic recovery of strain predominates. Reloading follows approximately the same path as the unloading. Finally, to model the reloading along a different path (from the unloading) a visco-elastic strain is introduced, whose rate of recovery is inversely proportional to the stress level. (The introduction of visco-elasticity in this form is only valid for uniaxial stress conditions. For multiaxial loading, the components in each component direction would need to be included.) The rate of visco-elastic strain is integrated into the combined model in the form

$$\varepsilon^v = a\sigma - \frac{b\varepsilon^v}{\sigma} \tag{133}$$

and setting

$$a = 0.01 \quad \text{and} \quad b = 5$$

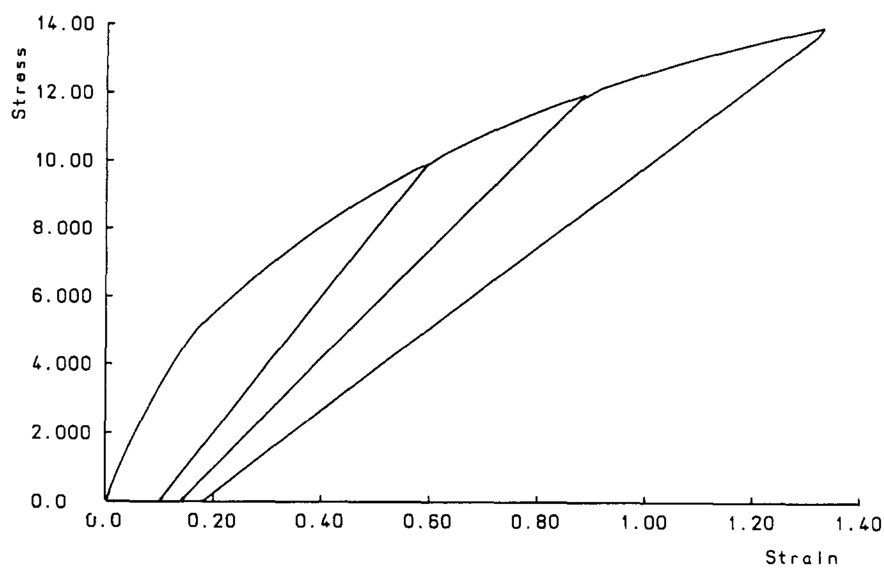
which results in an increase in visco strain recovery at lower stress levels with an infinite recovery at zero stress. Reloading initially occurs along an essentially elastic path as all the visco-elastic strain will be recovered once the stress reduces to zero. Figure 4(c) illustrates this behaviour. Very small steps were taken as the stress approached zero. To overcome the infinity inherent in the visco-elastic definition (133), the stress was reduced to 0.001 instead of zero. It is noted that a stand-alone program was written to explicitly integrate the rate equations. 20,000 steps were taken so that the solution is considered to be a target solution. The crosses in Figs 4(b) and 4(c) indicate solution points from the corresponding implicit integration using the uniaxial formulation. In general, the implicit solution performed well.

Example 5. A cylindrical shell subjected to self weight

The cylindrical shell shown in Fig. 5(a), is simply supported at both ends and free on its straight sides, and is subjected to a uniformly distributed self-weight load increasing to collapse. Taking advantage of symmetry the calculations are carried out for a quarter of the shell. The material is assumed to be elastic–perfectly plastic with the elastic property data being $E = 2.1\mathrm{E}7$ and $\nu = 0$. The Hoffman criterion is employed to describe the nonlinear anisotropy of the material with the uniaxial yield stresses:

$$\sigma^{t}_{11,y} = 4.3\mathrm{E}4, \quad \sigma^{c}_{11,y} = \sigma^{t}_{22,y} = \sigma^{c}_{22,y} = \sigma^{t}_{33,y} = \sigma^{c}_{33,y} = 4.3\mathrm{E}3,$$
$$\sigma_{12,y} = \sigma_{23,y} = \sigma_{31,y} = 2.4\mathrm{E}3.$$

(a)



(b)

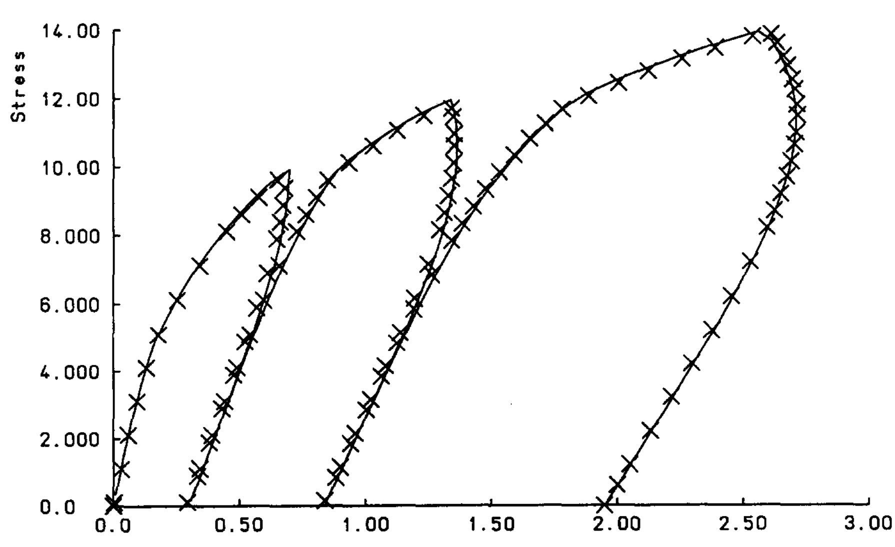


Fig. 4(a). Modelling of polymer type behaviour. Stage I: combination of damage and time-independent plastic flow. (b) Modelling of polymer type behaviour. Stage II: combination of damage and time-dependent and time-independent plastic flow. *Legend*: target ——; LUSAS uniaxial algorithm (Section 5) ×.

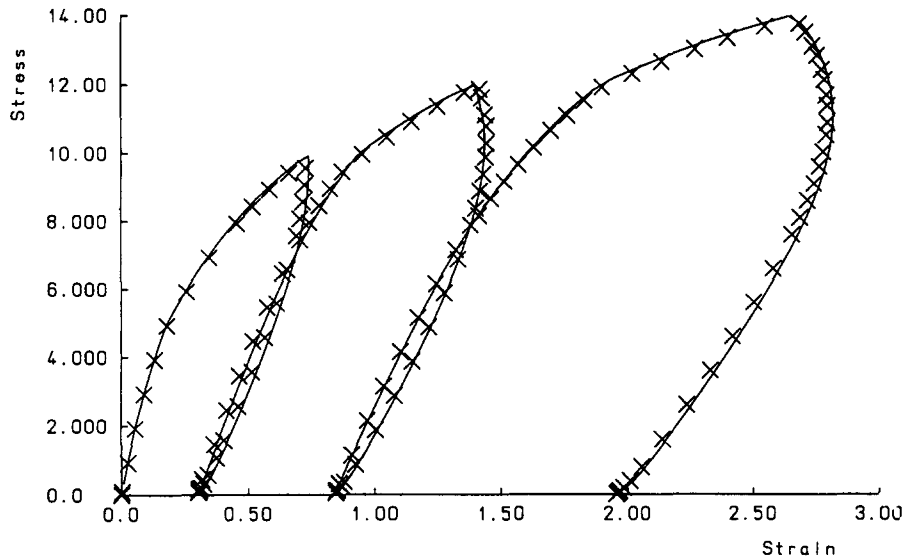


Fig. 4(c). Modelling of polymer type behaviour. Stage III: combination of damage and time-dependent and time-independent plastic flow and visco-elastic straining. *Legend*: target ——; LUSAS uniaxial algorithm (Section 5) ×.

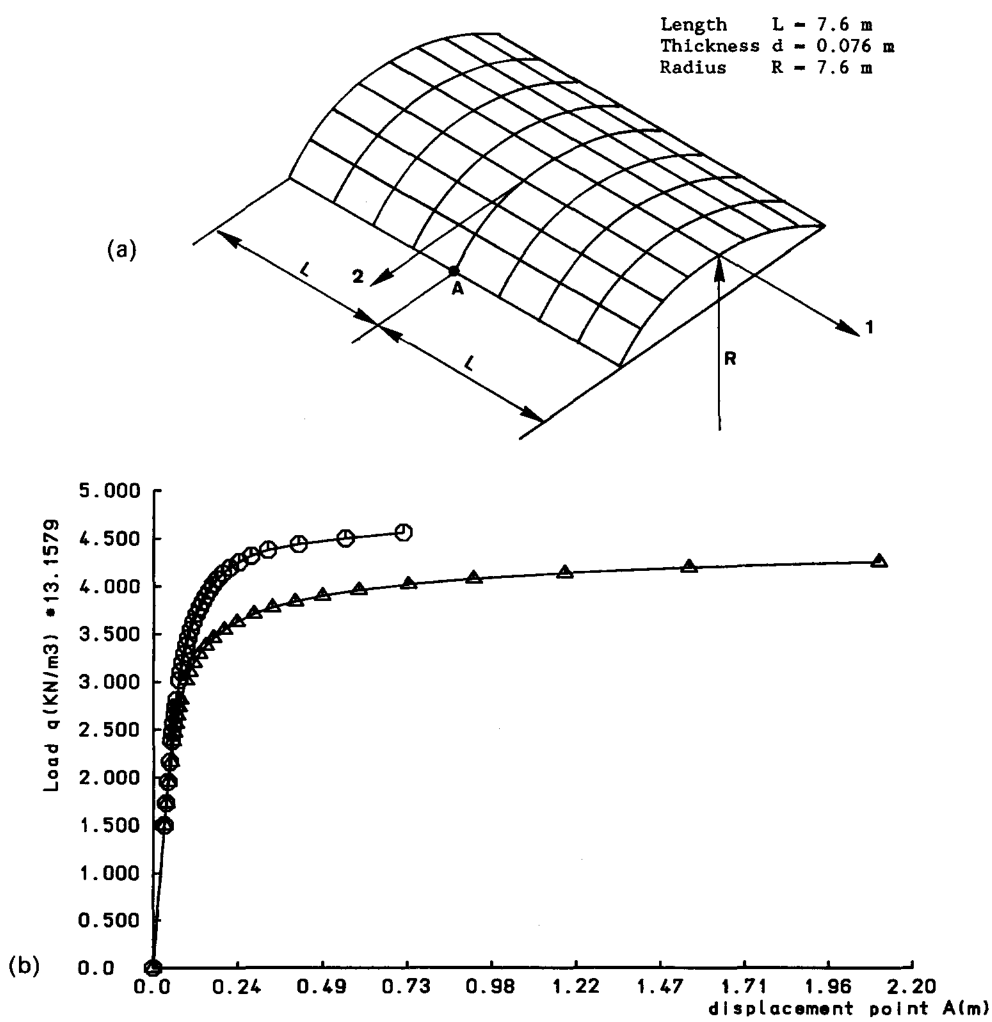


Fig. 5(a). Geometry of barrel shell subject to uniformly distributed load. (b) Failure of orthotropic barrel shell subject to uniformly distributed load. *Legend*: LUSAS without damage ⊕; LUSAS with damage △.

Simo's damage model is used with the damage property data ${}^0r = 0.8$, $A = B = 1.0$. A $4 \times 4 \times 1$ mesh of Semiloof elements is used to discretize the shell. Figure 5(b) shows the load–displacement curve with damage compared to that obtained without damage. The inclusion of damage results in a reduction of the total load carrying capacity from 4.6 KN to 4.1 KN.

9. CONCLUSIONS

A versatile nonlinear material model has been derived which is capable of efficiently describing time dependent, and time independent plastic flow as well as the degradation of material strength defined by an isotropic damage model. The formulation allows the use of each element uniquely, or in combination with the others, to cover a wide range of engineering materials. A polymer example was presented to illustrate how the combination of these elements can reproduce quite complex material behaviour.

The isotropic models take on exceptionally simple forms which makes them computationally very efficient. The extension of these models in particular, to include additionally effects such as hysteresis and visco-elasticity should present few obstacles, whilst retaining a high degree of numerical efficiency.

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